Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 326

Probability & Statistics for Engineers
Exam #2

Wednesday January 14, 2009

Duration: 55 minutes

Name: ______

Section: [illegible]

Instructor: [illegible]

Grade: [illegible]

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Please note that you have 5 questions and 6pages

Round all your answers to 3 digits after the decimal point.

1) **(16 points)** Suppose that 25% of NDU students are Females. 100 students were randomly selected. Find the probability that at least 20 of them are Females.

$p = 0.25 \Rightarrow q = 0.75$

let X be a R.V to represent # of female student selected among 100

$P(X \geq 20)$ ∴ X: binomial

$n = 100$.

since n is large, we replace the binomial variable X by a normal continuous R.V Y.

$\Rightarrow P(X \geq 20) = P(Y \geq 19.5)$

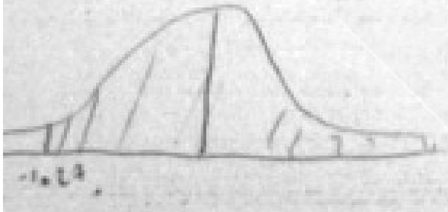
and $\frac{X - \mu}{\delta} = Z$ with $\mu_Y = np = 100 \times 0.25 = 25$

$\delta_Y = \sqrt{npq} = 4.33$.

$\Rightarrow P(Y \geq 19.5) = P\left(Z \geq \frac{19.5 - 25}{4.33}\right) = P(Z \geq -1.27)$

$= 0.5 + A_{1.27} = 0.5 \pm 0.398 = 0.102$

0.898



-1.27

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2) (12 points) The resistances of a certain type of resistors are independent with a mean of 50 Ohms and a standard deviation of 4 Ohms. Find the probability that the total resistance of 36 randomly selected of such resistors connected in series is less than or equal to 1818 Ohms.

$\mu = 50$

$\sigma = 4$

$n = 36$

let $X_1, X_2, \dots X_{36}$ be R.Vs to represent the resistance of resistors 1, 2, ... 36

$\Rightarrow P\left(\frac{X_1 + X_2 + \dots X_{36}}{36} \le \frac{1818}{36} \right)$

$\Rightarrow P(\bar{X} \le 50.5)$.

since n is large, we use the Z distribution!

$\Rightarrow P\left(\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \le \frac{50.5 - \mu}{\sigma / \sqrt{n}} \right) \Rightarrow P\left(Z \le \frac{50.5 - 50}{4 / \sqrt{36}} \right)$

$\Rightarrow P(Z \le 0.75) = 0.5 + A_{0.75} = 0.5 + 0.2734 = 0.7734$.

3) **(21 points)** Acid gases must be removed from other refinery gases in chemical production facilities to minimize corrosion rate of the planets. Two methods can be used for removing acid gases. For method (1), a random sample of size 7 shows the following corrosion rates: 0.48 , 0.65 , 0.8 , 0.9 , 0.5 , 0.8 , 0.7

For method (2), an independent random sample of size 7 gave us $\bar{x}_2 = 0.72$ and $s_2 = 0.11$.

$s^2 = \frac{1}{n-1}\left[\sum x_i^2 - n\bar{x}^2\right]$

a) Construct a 90% confidence interval for the difference between the true mean corrosion rates of these two methods, assuming that the two populations are normal with unknown but equal variances.

$\mathrm{I}: \begin{cases} n_1 = 7 \\ \bar{x}_1 = \frac{\sum x_i}{n} = 0.69 \\ s_1 = 0.158 \end{cases}$ $\quad \mathrm{II}: \begin{cases} n_2 = 7 \\ \bar{x}_2 = 0.72 \\ s_2 = 0.11 \end{cases}$ $\quad df = n_1 + n_2 - 2 = 12$

$1 - \alpha = 0.9 \Rightarrow \frac{\alpha}{2} = 0.05$

Since n_1 and n_2 are small and σ not given and the populations are normal with equal variances $\Rightarrow$ we use t-dist $\Rightarrow \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1-\mu_2)}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$

and the 90% confidence interval for $\mu_1 - \mu_2 =$

$\left[(\bar{x}_1 - \bar{x}_2) - t_{\frac{\alpha}{2}}\, s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \; ; \; \bar{x}_1 - \bar{x}_2 + t_{\frac{\alpha}{2}}\, s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right]$

with $s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{6(0.158)^2 + 6(0.11)^2}{12}} = 0.1179$

$= 0.136$

and $t_{0.05}(12) = 1.782$

$\Rightarrow \left[(0.69 - 0.72) - 1.782 \times 0.1179\sqrt{\tfrac{2}{7}} \; ; \; (0.69 - 0.72) + 1.782 \times 0.1179\ldots\right.$

$= \left[-0.1422 \; ; \; 0.0822\right] \quad \left[-0.159, 0.09\ldots\right.$

b) Based on your confidence interval, can we conclude that the mean corrosion rate for method (2) is higher than that of method (1)? Explain.

yes because $\frac{-0.1422 + 0.0822}{2} = -0.03 < 0$

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$\Rightarrow \mu_1 - \mu_2 < 0$

$\mu_2 - \mu_1 > 0$

$\mu_2 > \mu_1$

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4) (30 points) The joint probability density function of the two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{3}(x+y) & \text{for } 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere.} \end{cases}$$

a) Find $E(X)$ and $E(Y)$.

$$f_1(x) = \frac{1}{3}\int_0^2 (x+y)\,dy = \frac{1}{3}\left(xy + \frac{1}{2}y^2\Big|_0^2\right) = \frac{1}{3}(2x+2) = \begin{cases} \frac{2}{3}(x+1) & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_2(y) = \frac{1}{3}\int_0^1 (x+y)\,dx = \frac{1}{3}\left(\frac{x^2}{2} + yx\Big|_0^1\right) = \frac{1}{3}\left(\frac{1}{2}+y\right)$$

$$= \begin{cases} \frac{1}{3}\left(y+\frac{1}{2}\right) & 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\cdot\ E(X) = \int_0^1 x f_1(x)\,dx = \frac{2}{3}\int_0^1 (x^2+x)\,dx = \frac{2}{3}\left(\frac{x^3}{3}+\frac{x^2}{2}\Big|_0^1\right) = \frac{2}{3}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{2}{3}\left(\frac{5}{6}\right)$$

$$= \boxed{\frac{5}{9}}$$

$$\cdot\ E(Y) = \int_0^2 y f_2(y)\,dy = \frac{1}{3}\int_0^2 \left(y^2 + \frac{1}{2}y\right)dy = \frac{1}{3}\left(\frac{y^3}{3} + \frac{y^2}{4}\Big|_0^2\right) = \frac{1}{3}\left(\frac{8}{3}+1\right)$$

$$= \frac{20}{3}\times\frac{1}{3} = \boxed{\frac{20}{9}}$$

$\frac{11}{9}$

b) Find $Cov(X,Y)$

$$Cov(X,Y) = E(X,Y) - E(X)\cdot E(Y)$$

$$\cdot\ E(X,Y) = \frac{1}{3}\int_0^2\int_0^1 xy\,(x+y)\,dx\,dy = \frac{1}{3}\int_0^2\int_0^1 (x^2y + xy^2)\,dx\,dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2\Big|_0^1\right)dy = \frac{1}{3}\int_0^2\left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \frac{1}{3}\left(\frac{y^2}{6}+\frac{y^3}{6}\Big|_0^2\right)$$

$$5\quad = \frac{1}{3}\left(\frac{4}{6}+\frac{8}{6}\right) = \frac{1}{3}\left(\frac{12}{6}\right) = \boxed{\frac{2}{3}}$$

$Cov(X,Y) = ?$

c) Are X and Y independent? Give reasons.

$f_1(x) \cdot f_2(y) = \frac{2}{3}(x+1) \times \frac{1}{3}(y+1)$

$= \frac{1}{9}(xy + x + y)$

Since $f(x,y) \neq f_1(x) \cdot f_2(y) \Rightarrow X$ and Y are dependent.

d) Find $P\left(X \le \frac{1}{2} \middle| Y = \frac{1}{2}\right)$.

$= \dfrac{P(-\infty \le X \le \frac{1}{2}, Y = \frac{1}{2})}{P(Y = \frac{1}{2})} = \dfrac{\int_{-\infty}^{1/2} f(x, \frac{1}{2})\, dx}{f_2(\frac{1}{2})}$

$= \dfrac{\frac{1}{3}\int_{0}^{1/2}(x + \frac{1}{2})\, dx}{\frac{1}{3}(\frac{1}{2}+1)} = \dfrac{\frac{1}{3}\left|\frac{x^2}{2} + \frac{1}{2}x\right|_0^{1/2}}{\frac{1}{3}(\frac{3}{2})}$

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5) (21 points) Customers often complain about the long waiting time at restaurants before the food is served. New restaurant claims that its customers usually be served in maximum 15 minutes after the order is placed. To check the claim a random sample of 49 customers shows that the mean waiting time taken to serve them is 16 minutes, with a standard deviation of 3.2 minutes.

a) At $\alpha = 0.05$ is there sufficient evidence to reject the claim?

$n = 49$, $\bar{x} = 16$, $s = 3.2$

$H_0: \mu \leq 15$

$H_a: \mu > 15$.

since n is large, we use Z distribution

$\frac{\bar{x} - \mu_{\bar{x}}}{s_{\bar{x}}} = Z$

Right sided test:

acceptance region | rejection region

$Z_\alpha = 0.0199 = 1.645$

$Z_{0.05} = 0.0199$

$$O.V.T.S = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{16 - 15}{3.2/\sqrt{49}} = 2.1875.$$

The O.V.T.S is in the rejection region $\Rightarrow$ we reject the claim.

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b) Find the p-value of the test. Based on this p-value, would you accept the claim at $\alpha = 0.01$? Explain.

$P\text{-value} = P(Z \geq O.V.T.S) = P(Z > 2.1875) = 0 < 0.01$

$\alpha > p\text{-value} \Rightarrow$ we reject the claim.

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